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The Searchlight Problem for Radiative Transfer in a Finite Slab

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A version of the discrete-ordinates method recently developed for radiative-transfer calculations is used along with numerical linear-algebra techniques and two-dimensional Fourier-transform procedures to establish the radiation flux and the z component of the radiation current at all locations in a finite plane-parallel layer irradiated by a beam incident only at one point on one surface. In addition to a general formulation basic to a beam that is incident at an oblique angle, for which the flux and current depend on three spatial variables, the Fourier transforms of the flux and current are inverted numerically for the two-dimensional case relevant to a normally incident beam. The reported numerical procedures, while computationally intensive, are thought to yield, for the considered test case, the radiation flux and the normal component of the radiation current with five figures of accuracy.

1. INTRODUCTION

One has to admit that the classical searchlight problem defined in the field of radiative transfer by Chandrasekhar [1] some 40 years ago still today represents a problem in particle-transport theory that is sufficiently difficult that very few high-quality computational results have been reported. In regard to early papers devoted to problems somewhat related to the searchlight problem, we consider that Elliott [2], who based his analysis on two-dimensional Fourier-transform procedures, defined the approach that has led to the (limited quantity of) semi-analytical results in existence today. We note that Rybicki [3] has given an extensive review of early work devoted explicitly to the searchlight problem. As for more recent efforts, we can say, to the best of our knowledge, that Refs. [4–8] are the ones most directly related to our work here. As did Elliott [2], we use two-dimensional Fourier-transform techniques, and while much can be done in transform space, we consider that without the evaluation of the required inversion integrals the job is in no way complete. It is for this reason that we consider Refs. [6, 8–10] to be particularly significant.

The solution developed here has very much in common with Ref. [8], but instead of basing our solution to a certain "pseudo problem" on the F_N method [11] we make use of

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some recent work [12, 13] with the discrete-ordinates method in order to provide a new, alternative solution.

The searchlight problem we consider is defined by the equation of transfer

$$\mu \frac{\partial}{\partial z} I(z, \boldsymbol{\rho}, \boldsymbol{\Omega}) + \boldsymbol{\omega} \cdot \frac{\partial}{\partial \boldsymbol{\rho}} I(z, \boldsymbol{\rho}, \boldsymbol{\Omega}) + I(z, \boldsymbol{\rho}, \boldsymbol{\Omega}) = \frac{\varpi}{4\pi} \iint I(z, \boldsymbol{\rho}, \boldsymbol{\Omega}') \, \mathrm{d}\boldsymbol{\Omega}', \quad (1)$$

for all z, $\boldsymbol{\rho}$, and $\boldsymbol{\Omega}$, and the boundary conditions

$$I[0, \boldsymbol{\rho}, \boldsymbol{\Omega}(\mu, \phi)] = \frac{1}{2\pi\rho} \delta(\rho)\delta(\mu - \mu_0)\delta(\phi - \phi_0) \quad (2a)$$

and

$$I[z_0, \boldsymbol{\rho}, \boldsymbol{\Omega}(-\mu, \phi)] = 0 \quad (2b)$$

for $\mu \in (0, 1]$ and $\phi \in [0, 2\pi]$. We follow closely the notation of Rybicki [3] and note that $z \in [0, z_0]$ and $\boldsymbol{\rho}$, which lies in the x-y plane, locate (in terms of mean free paths) the position in the layer and that $\boldsymbol{\Omega}(\mu, \phi)$, with $\mu = \cos\theta$, is a unit vector that defines the direction of propagation (see Fig. 1). In addition, $\boldsymbol{\omega}$ is the projection of $\boldsymbol{\Omega}$ in the x-y plane, $\boldsymbol{\Omega}(\mu_0, \phi_0)$ defines the direction of the incident beam, and $\varpi < 1$ is the mean number of secondary particles per collision.

Considering that Eqs. (1) and (2) define our basic problem, we seek to compute the radiation flux

$$I_0(z, \boldsymbol{\rho}) = \iint I(z, \boldsymbol{\rho}, \boldsymbol{\Omega}) \, \mathrm{d}\boldsymbol{\Omega} \quad (3a)$$

and the z component of the radiation current

$$I_1(z, \boldsymbol{\rho}) = \iint \mu I(z, \boldsymbol{\rho}, \boldsymbol{\Omega}) \, \mathrm{d}\boldsymbol{\Omega} \quad (3b)$$

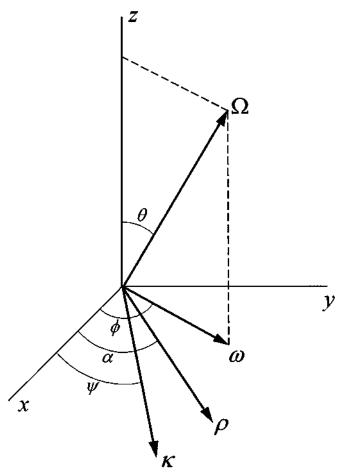

FIGURE 1

for $z \in [0, z_0]$ and all $\boldsymbol{\rho}$ of interest. Continuing to follow previously mentioned works, we let $\boldsymbol{k}$, given in terms of $k = |\boldsymbol{k}|$ and ψ (see Fig. 1), define our transform vector so that we can take a two-dimensional Fourier transform of Eqs. (1) and (2) to find the transfer equation (in transform space)

$$\mu\frac{\partial}{\partial z}\Psi(z, \mu, \phi : \boldsymbol{k}) + u(\mu, \phi : \boldsymbol{k})\Psi(z, \mu, \phi : \boldsymbol{k}) = \frac{\varpi}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi}\Psi(z, \mu', \phi' : \boldsymbol{k})\,\mathrm{d}\phi'\,\mathrm{d}\mu', \tag{4}$$

for $z \in (0, z_0)$, $\mu \in [-1, 1]$ and all ϕ, and the boundary conditions

$$\Psi(0, \mu, \phi : \boldsymbol{k}) = \delta(\mu - \mu_0)\delta(\phi - \phi_0) \tag{5a}$$

and

$$\Psi(z_0, -\mu, \phi : \boldsymbol{k}) = 0 \tag{5b}$$

for $\mu \in (0, 1]$ and $\phi \in [0, 2\pi]$. Here

$$u(\mu, \phi : \boldsymbol{k}) = 1 - ik(1 - \mu^2)^{1/2}\cos(\phi - \psi) \tag{6}$$

and

$$\Psi(z, \mu, \phi : \boldsymbol{k}) = \iint I(z, \boldsymbol{\rho}, \boldsymbol{\Omega})\exp\{i\boldsymbol{k}\cdot\boldsymbol{\rho}\}\,\mathrm{d}\boldsymbol{\rho}. \tag{7}$$

We can also take the Fourier transform of Eqs. (3) to obtain

$$\iint I_0(z, \boldsymbol{\rho})\exp\{i\boldsymbol{k}\cdot\boldsymbol{\rho}\}\,\mathrm{d}\boldsymbol{\rho} = \Psi_0(z : \boldsymbol{k}) \tag{8a}$$

and

$$\iint I_1(z, \boldsymbol{\rho})\exp\{i\boldsymbol{k}\cdot\boldsymbol{\rho}\}\,\mathrm{d}\boldsymbol{\rho} = \Psi_1(z : \boldsymbol{k}), \tag{8b}$$

where

$$\Psi_0(z : \boldsymbol{k}) = \int_{-1}^{1}\int_{0}^{2\pi}\Psi(z, \mu, \phi : \boldsymbol{k})\,\mathrm{d}\phi\,\mathrm{d}\mu \tag{9a}$$

and

$$\Psi_1(z : \boldsymbol{k}) = \int_{-1}^{1}\int_{0}^{2\pi}\mu\Psi(z, \mu, \phi : \boldsymbol{k})\,\mathrm{d}\phi\,\mathrm{d}\mu. \tag{9b}$$

Of course once we have $\Psi_0(z : \boldsymbol{k})$ and $\Psi_1(z : \boldsymbol{k})$ the radiation flux and the z component of the radiation current are available, at least in principle, from the inversion integrals

$$I_0(z, \boldsymbol{\rho}) = \frac{1}{(2\pi)^2}\iint \Psi_0(z : \boldsymbol{k})\exp\{-i\boldsymbol{k}\cdot\boldsymbol{\rho}\}\,\mathrm{d}\boldsymbol{k} \tag{10a}$$

and

$$I_1(z, \boldsymbol{\rho}) = \frac{1}{(2\pi)^2} \iint \Psi_1(z : \boldsymbol{k}) \exp\{-i\boldsymbol{k} \cdot \boldsymbol{\rho}\} \, d\boldsymbol{k} \tag{10b}$$

or

$$I_0(z, \rho, \alpha) = \frac{1}{(2\pi)^2} \int_0^\infty \int_0^{2\pi} k\Psi_0(z : \boldsymbol{k}) \exp\{-ik\rho\cos(\alpha - \psi)\} \, d\psi \, dk \tag{11a}$$

and

$$I_1(z, \rho, \alpha) = \frac{1}{(2\pi)^2} \int_0^\infty \int_0^{2\pi} k\Psi_1(z : \boldsymbol{k}) \exp\{-ik\rho\cos(\alpha - \psi)\} \, d\psi \, dk. \tag{11b}$$

And so it is clear that to obtain the radiation flux and current we should compute $\Psi_0(z : \boldsymbol{k})$ and $\Psi_1(z : \boldsymbol{k})$ and then evaluate the inversion integrals given by Eqs. (11).

2. THE PSEUDO PROBLEM

Rather than trying to find $\Psi_0(z : \boldsymbol{k})$ and $\Psi_1(z : \boldsymbol{k})$ directly from the defining Eqs. (4) and (5), we proceed as was done in previous works [4–6, 8] and base our analysis on the "pseudo problem" that was used by Williams [14–16]. In order to see well the connection between the problem defined by Eqs. (4) and (5) and Williams' reduced problem, we deduce, first of all, from Eqs. (4) and (5) that

$$\Psi(z, \mu, \phi : \boldsymbol{k}) = \Psi_*(z, \mu, \phi : \boldsymbol{k}) + \frac{\varpi}{4\pi\mu} \int_0^z \Psi_0(z' : \boldsymbol{k}) \exp\{-(z - z')/U(\mu, \phi : \boldsymbol{k})\} \, dz' \tag{12a}$$

and

$$\Psi(z, -\mu, \phi : \boldsymbol{k}) = \frac{\varpi}{4\pi\mu} \int_z^{z_0} \Psi_0(z' : \boldsymbol{k}) \exp\{-(z' - z)/U(\mu, \phi : \boldsymbol{k})\} \, dz' \tag{12b}$$

for $\mu \in (0, 1]$ and all ϕ. Here

$$\Psi_*(z, \mu, \phi : \boldsymbol{k}) = \delta(\mu - \mu_0)\delta(\phi - \phi_0) \exp\{-z/U(\mu, \phi : \boldsymbol{k})\}, \tag{13}$$

with

$$U(\mu, \phi : \boldsymbol{k}) = \mu/u(\mu, \phi : \boldsymbol{k}). \tag{14}$$

At this point we can follow Ref. [8] and integrate Eqs. (12) over μ and ϕ and add the resulting two equations to obtain the integral equation

$$\Psi_0(z : \boldsymbol{k}) = F(z : \boldsymbol{k}) + \frac{\varpi}{2} \int_0^{z_0} K(|z' - z| : k)\Psi_0(z' : \boldsymbol{k}) \, dz', \tag{15}$$

where $\Psi_0(z : \boldsymbol{k})$ is defined by Eq. (9a) and

$$F(z : \boldsymbol{k}) = \exp\{-z/U_0(\boldsymbol{k})\} \tag{16}$$

with

$$U_0(\boldsymbol{k}) = \mu_0/u(\mu_0, \phi_0 : \boldsymbol{k}). \tag{17}$$

Here the kernel of the integral equation is

$$K(\xi : k) = \int_0^1 (1 + k^2\mu^2)^{-1/2} \exp\left\{-\frac{\xi}{\mu}(1 + k^2\mu^2)^{1/2}\right\} \frac{\mathrm{d}\mu}{\mu}. \tag{18}$$

We note that the derivation of Eq. (15) is somewhat involved, but some helpful details relevant to this computation are given in Ref. [8].

While Eq. (15) was derived from Eqs. (4) and (5), it was observed by Williams [14–16] that the same integral equation can be obtained from what we call a pseudo problem. This problem is defined by the equation of transfer

$$\begin{aligned} &\mu(1 + k^2\mu^2)^{1/2} \frac{\partial}{\partial z}\Phi(z, \mu : \boldsymbol{k}) + (1 + k^2\mu^2)\Phi(z, \mu : \boldsymbol{k}) \\ &\quad = \frac{\varpi}{2}\int_{-1}^1 \Phi(z, \mu' : \boldsymbol{k})\,\mathrm{d}\mu' + \frac{1}{2}F(z : \boldsymbol{k}), \end{aligned} \tag{19}$$

for $\mu \in [-1, 1]$ and $z \in (0, z_0)$, and the boundary conditions

$$\Phi(0, \mu : \boldsymbol{k}) = 0 \tag{20a}$$

and

$$\Phi(z_0, -\mu : \boldsymbol{k}) = 0 \tag{20b}$$

for $\mu \in (0, 1]$. We can now follow the same procedure we used to develop Eq. (15) to obtain from Eqs. (19) and (20) the integral equation

$$\Phi(z : \boldsymbol{k}) = \frac{1}{2}\int_0^{z_0} K(|z' - z| : k)[\varpi\Phi(z' : \boldsymbol{k}) + F(z' : \boldsymbol{k})]\,\mathrm{d}z', \tag{21}$$

and so it follows that we can use

$$\Psi_0(z : \boldsymbol{k}) = F(z : \boldsymbol{k}) + \varpi\Phi(z : \boldsymbol{k}), \tag{22}$$

where

$$\Phi(z : \boldsymbol{k}) = \int_{-1}^1 \Phi(z, \mu : \boldsymbol{k})\,\mathrm{d}\mu, \tag{23}$$

to evaluate the right-hand side of Eq. (11a). Now, as discussed in Ref. [8], we can multiply Eqs. (12) by μ, integrate over μ and ϕ, and subtract one of the resulting equations from the other to find, after noting Eq. (9b),

$$\Psi_1(z : \boldsymbol{k}) = \mu_0 F(z : \boldsymbol{k}) + \frac{\varpi}{2}\int_0^{z_0} \mathrm{sgn}(z - z')M(|z' - z| : k)\Psi_0(z' : \boldsymbol{k})\,\mathrm{d}z', \tag{24}$$

where

$$M(\xi : k) = \int_0^1 \exp\left\{-\frac{\xi}{\mu}(1 + k^2\mu^2)^{1/2}\right\} d\mu. \tag{25}$$

Defining

$$\Xi(z : \boldsymbol{k}) = \int_{-1}^1 \mu(1 + k^2\mu^2)^{1/2}\Phi(z, \mu : \boldsymbol{k})\, d\mu, \tag{26}$$

we find from Eqs. (19) and (20) that

$$\Xi(z : \boldsymbol{k}) = \frac{1}{2}\int_0^{z_0} \operatorname{sgn}(z - z')M(|z' - z| : k)[\varpi\boldsymbol{\Phi}(z' : \boldsymbol{k}) + F(z' : \boldsymbol{k})]\, dz', \tag{27}$$

and so we conclude that

$$\Psi_1(z : \boldsymbol{k}) = \mu_0 F(z : \boldsymbol{k}) + \varpi\,\Xi(z : \boldsymbol{k}) \tag{28}$$

can be used to evaluate the right-hand side of Eq. (11b).

While it is very clear from Eqs. (1) and (2) that the uncollided component of $I(z, \boldsymbol{\rho}, \boldsymbol{\Omega})$ is singular (it is a generalized function, to be more precise), it is not so evident that the once-collided component of $I(z, \boldsymbol{\rho}, \boldsymbol{\Omega})$ also is, as was pointed out in Refs. [5–8], a generalized function. And so, as a result of the singular nature of these two components of $I(z, \boldsymbol{\rho}, \boldsymbol{\Omega})$, the desired solution, in earlier work [5–8], was split into three elements: the uncollided beam, the once-collided term, and the remainder. In fact, this decomposition was considered essential when an attempt was made to carry out a numerical evaluation of some required Fourier-transform inversion integrals. In this work we seek to compute the flux and the z component of the current and not the angular flux, and so this decomposition into three elements is not necessary here. It follows that our basic job now is to develop the required quantities that have been expressed in terms of the pseudo problem defined by Eqs. (16), (19), and (20). As we wish to make use of some recent work with the discrete-ordinates method we choose to rewrite Eqs. (19) and (20) in terms of new variables. If we let

$$(1 + k^2\mu^2)\Phi(z, \mu : \boldsymbol{k}) = \hat{\Phi}(z, \mu : \boldsymbol{k}), \tag{29a}$$

$$\mu = \xi(1 - k^2\xi^2)^{-1/2}, \tag{29b}$$

and

$$\hat{\Phi}[z, \xi(1 - k^2\xi^2)^{-1/2} : \boldsymbol{k}] = G(z, \xi : \boldsymbol{k}) \tag{29c}$$

then we can rewrite our problem as

$$\xi\frac{\partial}{\partial z}G(z, \xi : \boldsymbol{k}) + G(z, \xi : \boldsymbol{k}) = \frac{\varpi}{2}\int_{-\gamma}^{\gamma} \phi(u : k)G(z, u : \boldsymbol{k})\, du + \frac{1}{2}F(z : \boldsymbol{k}), \tag{30}$$

for $z \in (0, z_0)$ and $\xi \in [-\gamma, \gamma]$, with

$$G(0, \xi : \boldsymbol{k}) = 0 \tag{31a}$$

and

$$G(z_0, -\xi : \boldsymbol{k}) = 0 \tag{31b}$$

for $\xi \in (0, \gamma]$. Here

$$\gamma = (1 + k^2)^{-1/2} \tag{32a}$$

and

$$\phi(u : k) = (1 - k^2 u^2)^{-1/2}. \tag{32b}$$

Looking back to Eqs. (22) and (28), we can now write

$$\Psi_0(z : \boldsymbol{k}) = F(z : \boldsymbol{k}) + \varpi L(z : \boldsymbol{k}) \tag{33a}$$

and

$$\Psi_1(z : \boldsymbol{k}) = \mu_0 F(z : \boldsymbol{k}) + \varpi H(z : \boldsymbol{k}), \tag{33b}$$

where

$$L(z : \boldsymbol{k}) = \int_{-\gamma}^{\gamma} \phi(\xi : k) G(z, \xi : \boldsymbol{k})\, \mathrm{d}\xi \tag{34a}$$

and

$$H(z : \boldsymbol{k}) = \int_{-\gamma}^{\gamma} \xi \phi^3(\xi : \boldsymbol{k}) G(z, \xi : \boldsymbol{k})\, \mathrm{d}\xi. \tag{34b}$$

We see from Eqs. (11) and (33) that to complete our solution (at least in transform space) we must find a good way to compute the quantities $L(z : \boldsymbol{k})$ and $H(z : \boldsymbol{k})$. In Ref. [8] a version of the F_N method [11] was used for this purpose. Here we develop the required quantities in terms of a "half-range" discrete-ordinates solution of the "G problem" defined by Eqs. (30)–(32).

3. A DISCRETE-ORDINATES SOLUTION

It is clear that the G problem we must solve depends on the transform variable $\boldsymbol{k}$; however, in order to avoid too much heavy notation we choose, in this (and the next) section, to suppress the explicit $\boldsymbol{k}$-dependent notation we have used to this point in our work. We note from Eq. (16) that $F(z)$ is, in general, a complex function of a real variable, and so, of course, the resulting $G(z, \xi)$ is also a complex function. However, we can consider Eq. (30) written as

$$\xi \frac{\partial}{\partial z} G(z, \xi) + G(z, \xi) = \frac{\varpi}{2} \int_{-\gamma}^{\gamma} \phi(u) G(z, u)\, \mathrm{d}u + Q(z), \tag{35}$$

where we can take $Q(z)$ to be either the real or imaginary part of $F(z)/2$ and thus can obtain either the real or imaginary part of $G(z, \xi)$. Our development of a discrete-ordinates solution of Eq. (35) follows closely Ref. [12], and so we will be brief here. Since $\phi(u)$ is an even function, we write our discrete-ordinates equations as

$$\xi_i \frac{\mathrm{d}}{\mathrm{dz}} G(z, \xi_i) + G(z, \xi_i) = \frac{\varpi}{2} \sum_{k=1}^{N} w_k \phi(\xi_k)[G(z, \xi_k) + G(z, -\xi_k)] + Q(z) \quad (36a)$$

and

$$-\xi_i \frac{\mathrm{d}}{\mathrm{dz}} G(z, -\xi_i) + G(z, -\xi_i) = \frac{\varpi}{2} \sum_{k=1}^{N} w_k \phi(\xi_k)[G(z, \xi_k) + G(z, -\xi_k)] + Q(z) \quad (36b)$$

for $i = 1, 2, \ldots, N$. In writing Eqs. (36) as we have, we clearly are considering that the N quadrature points $\{\xi_k\}$ and the N weights $\{w_k\}$ are defined for use on the integration interval $[0, \gamma]$. Of course we are free to use a single quadrature scheme on the interval $[0, \gamma]$, or we can use a composite quadrature defined over sub-intervals of $[0, \gamma]$. Now seeking exponential solutions of the homogeneous equations, we substitute

$$G(z, \pm\xi_i) = \phi(\nu, \pm\xi_i) \exp\{-z/\nu\} \quad (37)$$

into the homogeneous versions of Eqs. (36) to find

$$\frac{1}{\nu} \boldsymbol{\Xi}\boldsymbol{\Phi}_+ = (\boldsymbol{I} - \boldsymbol{W})\boldsymbol{\Phi}_+ - \boldsymbol{W}\boldsymbol{\Phi}_- \quad (38a)$$

and

$$-\frac{1}{\nu} \boldsymbol{\Xi}\boldsymbol{\Phi}_- = (\boldsymbol{I} - \boldsymbol{W})\boldsymbol{\Phi}_- - \boldsymbol{W}\boldsymbol{\Phi}_+, \quad (38b)$$

where $\boldsymbol{I}$ is the $N \times N$ identity matrix,

$$\boldsymbol{\Phi}_\pm = [\phi(\nu, \pm\xi_1), \phi(\nu, \pm\xi_2), \ldots, \phi(\nu, \pm\xi_N)]^{\mathrm{T}}, \quad (39)$$

the elements of the $\boldsymbol{W}$ matrix are given by

$$(\boldsymbol{W})_{i,j} = \frac{\varpi}{2} w_j \phi(\xi_j) \quad (40)$$

and

$$\boldsymbol{\Xi} = \mathrm{diag}\{\xi_1, \xi_2, \ldots, \xi_N\}. \quad (41)$$

If we let

$$\boldsymbol{U} = \boldsymbol{\Phi}_+ + \boldsymbol{\Phi}_- \quad (42)$$

and

$$\boldsymbol{V} = \boldsymbol{\Phi}_+ - \boldsymbol{\Phi}_- \quad (43)$$

then we can eliminate $\boldsymbol{V}$ between the sum and the difference of Eqs. (38) to find

$$(\Xi^{-2} - 2\Xi^{-1}\boldsymbol{W}\Xi^{-1})\Xi\boldsymbol{U} = \frac{1}{\nu^2}\Xi\mathbf{U}, \tag{44}$$

where to have Ξ^{-1} exist we cannot allow any of the quadrature points to be zero. Multiplying Eq. (44) by the diagonal matrix $\boldsymbol{T}$ with the diagonal elements given by

$$T_i = \left[\frac{\varpi}{2} w_i \phi(\xi_i)\right]^{1/2} \tag{45}$$

we can make $\boldsymbol{TWT}^{-1}$ symmetric so we can rewrite Eq. (44) as

$$(\boldsymbol{D} - \mathbf{z}\mathbf{z}^T)\boldsymbol{X} = \lambda \boldsymbol{X}, \tag{46}$$

where

$$\boldsymbol{D} = \operatorname{diag}\{\xi_1^{-2}, \xi_2^{-2}, \ldots, \xi_N^{-2}\}, \tag{47}$$

$$\boldsymbol{X} = \boldsymbol{T}\Xi\mathbf{U}, \tag{48}$$

$$\mathbf{z} = \left[\frac{\sqrt{\varpi w_1 \phi(\xi_1)}}{\xi_1}, \frac{\sqrt{\varpi w_2 \phi(\xi_2)}}{\xi_2}, \ldots, \frac{\sqrt{\varpi w_N \phi(\xi_N)}}{\xi_N}\right]^{\mathrm{T}}, \tag{49}$$

and $\lambda = 1/\nu^2$. We note that the eigenvalue problem defined by Eq. (46) is of a form that is encountered when the so-called “divide and conquer” method [17] is used to find the eigenvalues of tridiagonal matrices. This special eigenvalue problem has been discussed in Ref. [18], and a special numerical package DZPACK has been made available [18] for dealing with this eigenvalue problem.

Considering that we have found the required eigenvalues from Eq. (46), we impose the normalization condition

$$\sum_{k=1}^{N} w_k \phi(\xi_k)[\phi(\nu, \xi_k) + \phi(\nu, -\xi_k)] = 1 \tag{50}$$

so that we can write our discrete-ordinates solution as

$$G_h(z, \pm\xi_i) = \sum_{j=1}^{N}[A_j \phi(\nu_j, \pm\xi_i)\exp\{-z/\nu_j\} + B_j \phi(\nu_j, \mp\xi_i)\exp\{-(z_0 - z)/\nu_j\}] \tag{51}$$

with

$$\phi(\nu_j, \pm\xi_i) = \frac{\varpi}{2}\frac{\nu_j}{\nu_j \mp \xi_i}. \tag{52}$$

The arbitrary constants $\{A_j\}$ and $\{B_j\}$ in Eq. (51) are to be determined from the boundary conditions, and the separation constants $\{\nu_j\}$ are the reciprocals of the positive square roots of the eigenvalues defined by Eq. (46). We note also that we have added the subscript h to remind us that the solution given by Eq. (51) applies only to the homogeneous versions of Eqs. (36).

Since Eqs. (36) have the inhomogeneous source term $Q(z)$ we must add a particular solution to $G_h(z, \pm\xi_i)$ to obtain the complete solution. Seeking a particular solution of the form

$$G_p(z, \pm\xi_i) = B(\pm\xi_i)\exp\{-z/U_0\}, \tag{53}$$

we find from Eqs. (36)

$$G_p(z, \pm\xi_i) = \frac{1}{2\Omega(U_0)}\frac{U_0}{U_0 \mp \xi_i}\exp\{-z/U_0\}, \tag{54}$$

where

$$\Omega(U_0) = 1 + \varpi\Gamma(U_0) \tag{55}$$

with

$$\Gamma(U_0) = U_0^2\sum_{\alpha=1}^{N}\frac{w_\alpha\phi(\xi_\alpha)}{\xi_\alpha^2 - U_0^2}. \tag{56}$$

We now write our complete solution as

$$G(z, \pm\xi_i) = G_h(z, \pm\xi_i) + G_p(z, \pm\xi_i), \tag{57}$$

where the arbitrary constants A_j and B_j required in Eq. (51) are defined by the system of linear algebraic equations obtained when we substitute Eq. (57) into Eqs. (31) evaluated at the quadrature points, viz.

$$\sum_{j=1}^{N}\left[A_j\phi(\nu_j, \xi_i) + B_j\phi(\nu_j, -\xi_i)e^{-z_0/\nu_j}\right] = -G_p(0, \xi_i) \tag{58a}$$

and

$$\sum_{j=1}^{N}\left[B_j\phi(\nu_j, \xi_i) + A_j\phi(\nu_j, -\xi_i)e^{-z_0/\nu_j}\right] = -G_p(z_0, -\xi_i) \tag{58b}$$

for $i = 1, 2, \ldots, N$. Once we have solved Eqs. (58) we can use Eq. (57) in our discrete-ordinates versions of Eqs. (34), viz.

$$L(z) = \sum_{k=1}^{N} w_k\phi(\xi_k)[G(z, \xi_k) + G(z, -\xi_k)] \tag{59a}$$

and

$$H(z) = \sum_{k=1}^{N} w_k\xi_k\phi^3(\xi_k)[G(z, \xi_k) - G(z, -\xi_k)] \tag{59b}$$

to find

$$L(z) = \sum_{j=1}^{N} [A_j \exp\{-z/\nu_j\} + B_j \exp\{-(z_0 - z)/\nu_j\}] + L_p(z) \tag{60a}$$

and

$$H(z) = (1-\varpi K) \sum_{j=1}^{N} \nu_j \phi^2(\nu_j)[A_j \exp\{-z/\nu_j\} - B_j \exp\{-(z_0 - z)/\nu_j\}] + H_p(z), \tag{60b}$$

where

$$L_p(z) = -\frac{\Gamma(U_0)}{\Omega(U_0)} \exp\{-z/U_0\} \tag{61a}$$

and

$$H_p(z) = \frac{U_0[K + \Gamma(U_0)]}{\Omega(U_0)\left(k^2 U_0^2 - 1\right)} \exp\{-z/U_0\}. \tag{61b}$$

In addition

$$K = \sum_{k=1}^{N} w_k \phi^3(\xi_k). \tag{62}$$

We note that we typically have used $K = 1$ in Eqs. (60b) and (61b) which, strictly speaking, is true only if the quadrature scheme used in Eq. (62) evaluates K exactly.

4. POST PROCESSING

While Eqs. (60) and (61) provide expressions we can readily evaluate to find the functions $L(z)$ and $H(z)$ required to complete Eqs. (33), we can also use the idea [19] of "post processing" to define alternative results. If we use Eq. (59a) to replace the integral term in Eq. (35), we can then solve the resulting equation to find

$$G(z, -\xi) = \frac{1}{\xi} \int_z^{z_0} \left[\frac{\varpi}{2} L(x) + Q(x)\right] \exp\{-(x - z)/\xi\}\, \mathrm{d}x \tag{63a}$$

and

$$G(z, \xi) = \frac{1}{\xi} \int_0^{z} \left[\frac{\varpi}{2} L(x) + Q(x)\right] \exp\{-(z - x)/\xi\}\, \mathrm{d}x \tag{63b}$$

for all $\xi \in (0, \gamma]$. Since $L(z)$, as given by Eqs. (60a) and (61a), and $Q(z)$ are defined by exponentials, we can integrate Eqs. (63) to find

$$G(z, -\xi) = \frac{U_0}{2\Omega(U_0)} \exp\{-z/U_0\} S(z_0 - z : U_0, \xi) + \frac{\varpi}{2} \Upsilon(z, -\xi) \tag{64a}$$

and

$$G(z,\xi) = \frac{U_0}{2\Omega(U_0)} C(z:U_0,\xi) + \frac{\varpi}{2}\Upsilon(z,\xi), \tag{64b}$$

where

$$\Upsilon(z,-\xi) = \sum_{j=1}^{N} \nu_j [A_j \exp\{-z/\nu_j\} S(z_0 - z : \nu_j, \xi) + B_j C(z_0 - z : \nu_j, \xi)] \tag{65a}$$

and

$$\Upsilon(z,\xi) = \sum_{j=1}^{N} \nu_j [A_j C(z : \nu_j, \xi) + B_j \exp\{-(z_0 - z)/\nu_j\} S(z : \nu_j, \xi)]. \tag{65b}$$

In addition, the S and C functions are given by

$$S(z:x,y) = \frac{1 - \mathrm{e}^{-z/x}\mathrm{e}^{-z/y}}{x+y} \tag{66a}$$

and

$$C(z:x,y) = \frac{\mathrm{e}^{-z/x} - \mathrm{e}^{-z/y}}{x-y}. \tag{66b}$$

Now since Eqs. (64) are valid for continuous values of ξ they can be used in

$$L(z) = \int_0^{\gamma} \phi(\xi)[G(z,\xi) + G(z,-\xi)]\,\mathrm{d}\xi \tag{67a}$$

and

$$H(z) = \int_0^{\gamma} \xi\phi^3(\xi)[G(z,\xi) - G(z,-\xi)]\,\mathrm{d}\xi \tag{67b}$$

to yield post-processed results for $L(z)$ and $H(z)$. To evaluate Eqs. (67) we can, as suggested by Nelson [20], use different quadrature rules for $L(z)$ and $H(z)$, and so we can write

$$L(z) = \sum_{\alpha=1}^{N_a} a_\alpha \phi(\eta_\alpha)[G(z,\eta_\alpha) + G(z,-\eta_\alpha)] \tag{68a}$$

and

$$H(z) = \sum_{\alpha=1}^{N_b} b_\alpha \zeta_\alpha \phi^3(\zeta_\alpha)[G(z,\zeta_\alpha) - G(z,-\zeta_\alpha)], \tag{68b}$$

where the N_a weights and nodes $\{a_\alpha, \eta_\alpha\}$ and the N_b weights and nodes $\{b_\alpha, \zeta_\alpha\}$ define two quadrature rules for use on the integration interval $[0, \gamma]$. We consider Eqs. (68) to be viable alternatives to Eqs. (60) especially since we are free to use convenient quadrature schemes to evaluate these expressions. We postpone a discussion of the numerical aspects of our computation until Section 6 where we make some additional comments concerning the relative merits of Eqs. (60) and (68).

Considering that we have solved our problem in transform space, we proceed to carry out the necessary inversions to find the results we seek.

5. THE FOURIER INVERSION

Having defined our solution of the pseudo problem, we consider that $L(z:\boldsymbol{k})$ and $H(z:\boldsymbol{k})$, as defined by Eqs. (60) or Eqs. (68), are available, and so we are ready to evaluate the Fourier inversion integrals to find the radiation fluxes and the radiation currents we seek. Note that we now indicate (again) more explicitly the dependence of various quantities on the Fourier-transform variable $\boldsymbol{k}$. While our analysis to this point has been general, we now restrict our computation to the special case of normal incidence ($\mu_0 = 1$) where one of the inversion integrals can be evaluated analytically. Here, since $\mu_0 = 1$ and thus $U_0(\boldsymbol{k}) = 1$, the transforms $\Psi_0(z:\boldsymbol{k})$ and $\Psi_1(z:\boldsymbol{k})$ are independent of the angle ψ, and so we rewrite Eqs. (11) as

$$I_\beta(z,\rho) = \frac{1}{2\pi}\int_0^\infty \Psi_\beta(z:k)\,J_0(k\rho)k\,\mathrm{d}k \tag{69}$$

for $\beta = 0$ or 1. Of course, we are using $J_0(x)$ to denote the first Bessel function of the first kind [21], and we are replacing $\boldsymbol{k}$ with k in our notation. Continuing, we now rewrite Eqs. (33) as

$$\Psi_0(z:k) = \exp\{-z\} + \varpi L(z:k) \tag{70a}$$

and

$$\Psi_1(z:k) = \exp\{-z\} + \varpi H(z:k). \tag{70b}$$

Next we can substitute Eqs. (70) into Eq. (69) to find

$$I_0(z,\rho) = \frac{\delta(\rho)}{2\pi\rho}\exp\{-z\} + \frac{\varpi}{2\pi}J(z,\rho) \tag{71a}$$

and

$$I_1(z,\rho) = \frac{\delta(\rho)}{2\pi\rho}\exp\{-z\} + \frac{\varpi}{2\pi}F(z,\rho), \tag{71b}$$

where

$$J(z,\rho) = \int_0^\infty L(z:k)\,J_0(k\rho)k\,\mathrm{d}k \tag{72a}$$

and

$$F(z,\rho) = \int_0^\infty H(z:k)\,J_0(k\rho)k\,\mathrm{d}k. \tag{72b}$$

We note that the quantities $J(z,\rho)$ and $F(z,\rho)$, which we rewrite as

$$J(z,\rho) = \frac{1}{\rho^2}\int_0^\infty xL(z:x/\rho)\,J_0(x)\,\mathrm{d}x \tag{73a}$$

and

$$F(z,\rho) = \frac{1}{\rho^2}\int_0^\infty xH(z:x/\rho)\,J_0(x)\,\mathrm{d}x, \tag{73b}$$

have been expressed in terms of our solution to the G problem, i.e., in terms of $L(z:k)$ and $H(z:k)$, and so are considered available, if not yet evaluated. Before proceeding to our numerical work we note that Eqs. (71) can be used as Green's functions and thus are defined, in a sense, for all ρ; however, those equations make sense numerically only for $\rho > 0$.

6. NUMERICAL RESULTS

Before discussing some details concerning the way we have implemented our solution to the G problem, we comment on the way we evaluated Eqs. (73) once the functions $L(z:k)$ and $H(z:k)$ were available. Having found that the integrands in Eqs. (73) oscillate about zero, we followed the work of Longman [22] and made use of an Euler transformation [23] to evaluate these difficult integrals numerically. To review the mentioned Euler transformation, consider the infinite series

$$S = \sum_{n=0}^{\infty} (-1)^n V_n, \tag{74}$$

where $V_n > 0$ and $V_{n+1} < V_n$ for all n. The Euler transformation allows us to rewrite S as

$$S = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} (-1)^n \Delta^n V_0, \tag{75}$$

where

$$\Delta^0 V_n = V_n, \tag{76a}$$

$$\Delta V_n = V_{n+1} - V_n, \tag{76b}$$

$$\Delta^2 V_n = \Delta V_{n+1} - \Delta V_n, \tag{76c}$$

and, in general,

$$\Delta^{r+1} V_n = \Delta^r V_{n+1} - \Delta^r V_n. \tag{76d}$$

We choose to use the binomial coefficients $B(i, j)$, $j = 1, 2, \ldots, i+1$, to express the terms in Eq. (75) in the form

$$\Delta^n V_0 = \sum_{j=1}^{n+1} (-1)^{j-1} B(n, j) V_{n+1-j}. \tag{77}$$

It follows [22, 23] that if the series defined by Eq. (74) converges, then so does the series defined by Eq. (75). In addition, and importantly for us here, when Eqs. (74) and (75) are approximated by a finite number of terms, then the resulting sum from Eq. (75) can yield a much more accurate result.

To test this idea of Longman (based, of course, on the Euler transformation) we considered the test quantity

$$E = 1 - \int_0^{\infty} J_0(x)\, \mathrm{d}x. \tag{78}$$

We approximated the infinite integration interval in Eq. (78) by a finite number of subintervals defined by the zeros of $J_0(x)$ and used integrals (we used a Gauss–Legendre quadrature scheme for each subinterval) over these subintervals to define the V_n in Eq. (74). Using as many as, say, 50 of the defined subintervals, we could get $|E|$ from Eq. (74) to be no smaller than, say, 10^{-3}. But from Eq. (75) we found $|E|$ as small as 10^{-13} using this same number of subintervals (or fewer).

Focusing our attention now on some of the computational details concerning the implementation of our discrete-ordinates solution of the G problem defined by Eqs. (30) and (31), we consider that the first thing we must do is to define a quadrature scheme. In that regard, we consider it important to note that the formulation of our discrete-ordinates solution is essentially independent of the quadrature scheme to be used. The only two restrictions we have imposed are that the N quadrature points $\{\xi_k\}$ and the N weights $\{w_k\}$ must be defined for use on the integration interval $[0, \gamma]$ and, because of the way our basic eigenvalue problem is formulated, that we must exclude zero from the set of quadrature points. Of course to exclude zero from the quadrature set is not considered a serious restriction since typical Gauss quadrature schemes do not include the end points of the integration interval.

To see a "reason" for using the quadrature scheme we have used in this work, we consider an integral of a form suggested by a "half-range" version of the integral term in Eq. (30), viz.

$$I = \int_0^{\gamma} \phi(\xi : k) F(\xi)\, \mathrm{d}\xi, \tag{79}$$

which we immediately rewrite as

$$I = \gamma \int_0^{1} \phi(\gamma x : k) F(\gamma x)\, \mathrm{d}x. \tag{80}$$

Here, we note Eqs. (32) and write

$$\phi(\gamma x : k) = \frac{1}{(1 - r^2 x^2)^{1/2}}, \tag{81}$$

where

$$r = \frac{k}{(1 + k^2)^{1/2}}. \tag{82}$$

It is clear that as r approaches 1 (large values of k) a linear mapping of a Gauss–Legendre scheme onto the interval [0, 1] is not going to work well for evaluating our considered integral. And so we let $x = g(y)$, with $g(0) = 0$, and rewrite Eq. (80) as

$$I = \gamma \int_0^{u} \phi[\gamma g(y) : k] F[\gamma g(y)] g'(y)\, \mathrm{d}y, \tag{83}$$

where $g(u) = 1$. We now set

$$\gamma g'(y) = [1 - r^2 g(y)^2]^{1/2} \tag{84}$$

and solve for $g(y)$ and u to obtain

$$g(y) = \frac{1}{r} \sin\left(\frac{ry}{\gamma}\right) \qquad \text{and} \qquad u = \frac{\gamma}{r} \arcsin(r). \tag{85}$$

At this point we consider that we have N Gauss–Legendre weights and nodes $\{\hat{w}_\alpha, \hat{x}_\alpha\}$ that have been mapped (linearly) onto the integration interval [0, 1], and so we write our quadrature version of Eq. (79) as

$$I = \sum_{\alpha=1}^{N} w_\alpha F(\xi_\alpha), \tag{86}$$

where

$$w_\alpha = u\hat{w}_\alpha \tag{87a}$$

and

$$\xi_\alpha = \gamma g(u\hat{x}_\alpha). \tag{87b}$$

So our discrete-ordinates solution of the G problem is based on the N weights and nodes defined by Eqs. (87).

Having defined our quadrature scheme, we have used the package DZPACK [18] to find the eigenvalues $\{\lambda_j\}$ from the eigenvalue problem defined by Eq. (46). The required separation constants were then available as the reciprocals of the square roots of these eigenvalues. We then used the subroutines DGECO and DGESL from the LINPACK package [24] to solve the linear system defined by Eqs. (58), and so the solution to the G problem was considered established.

In order to test our discrete-ordinates solution to the classical searchlight problem, we consider the particular case solved by the F_N method in Ref. [8], viz. we consider the case of a beam normally incident ($\mu_0 = 1$) on a finite slab of optical thickness $z_0 = 1$ with $\varpi = 0.8$. Our results are given, with what we believe to be five figures of accuracy, in Tables I and II.

Of course, we have carried out numerous numerical experiments to establish the confidence we have in our discrete-ordinates solution, and in comparing Tables I and II with the same tables from Ref. [8], we have found only one case that differed by more than one unit in the fifth figure reported [8.8949(–2) vs 8.8956(–2)]. This difference was found for what we consider the most difficult case: the radiation current calculation for $\rho = 0.001$ and $z = 0.1$. In the process of evaluating the numerical results obtained from our FORTRAN implementation of the discrete-ordinates solution developed here, we saw again a situation that deserves comment. For the radiation current calculation, in contrast to the flux calculation, the result can be positive, negative, and, in fact, zero. And so there clearly will be certain values of the independent variables for which a numerical computation (carried out on a machine with a finite word length) can yield values for which none of the figures are correct. While we may think of these special cases as truly exceptional, we must at the same time not take all suggestions of achieved accuracy to be definitive statements.

In regard to the computation of the functions $L(z : k)$ and $H(z : k)$ required for the Fourier inversion, we note that the simpler representations given by Eqs. (60) worked very well for all but, say, the two smallest of the considered values of ρ. To have the results with five

TABLE I

The Radiation Flux $I_0(\eta z_0, \rho)$ for $z_0 = 1.0$, $\varpi = 0.8$, and $\mu_0 = 1.0$

ρ	$\eta = 0$	$\eta = 0.05$	$\eta = 0.1$	$\eta = 0.2$	$\eta = 0.5$	$\eta = 0.75$	$\eta = 1$
0.001	9.9687(1)	1.8940(2)	1.8083(2)	1.6396(2)	1.2164(2)	9.4723(1)	3.7008(1)
0.010	9.7637	1.7924(1)	1.7703(1)	1.6356(1)	1.2291(1)	9.5620	3.8185
0.100	8.4077(−1)	1.1361	1.3086	1.4089	1.1991	9.2630(−1)	4.2588(−1)
0.200	3.6477(−1)	4.4620(−1)	5.0743(−1)	5.7604(−1)	5.4481(−1)	4.2286(−1)	2.1656(−1)
0.400	1.3991(−1)	1.6123(−1)	1.7785(−1)	2.0195(−1)	2.1090(−1)	1.7022(−1)	1.0015(−1)
0.600	7.2196(−2)	8.1428(−2)	8.8471(−2)	9.9110(−2)	1.0661(−1)	8.9826(−2)	5.7469(−2)
0.800	4.2094(−2)	4.6948(−2)	5.0567(−2)	5.6058(−2)	6.0809(−2)	5.2919(−2)	3.5779(−2)
1.000	2.6269(−2)	2.9097(−2)	3.1167(−2)	3.4295(−2)	3.7280(−2)	3.3174(−2)	2.3298(−2)
1.200	1.7130(−2)	1.8887(−2)	2.0154(−2)	2.2058(−2)	2.3981(−2)	2.1669(−2)	1.5634(−2)
1.400	1.1523(−2)	1.2663(−2)	1.3476(−2)	1.4692(−2)	1.5964(−2)	1.4581(−2)	1.0729(−2)
1.600	7.9341(−3)	8.6977(−3)	9.2373(−3)	1.0041(−2)	1.0902(−2)	1.0036(−2)	7.4936(−3)
1.800	5.5627(−3)	6.0867(−3)	6.4543(−3)	7.0001(−3)	7.5946(−3)	7.0314(−3)	5.3094(−3)
2.000	3.9573(−3)	4.3237(−3)	4.5792(−3)	4.9575(−3)	5.3745(−3)	4.9978(−3)	3.8072(−3)
2.200	2.8490(−3)	3.1092(−3)	3.2896(−3)	3.5562(−3)	3.8527(−3)	3.5949(−3)	2.7577(−3)
2.400	2.0718(−3)	2.2587(−3)	2.3879(−3)	2.5782(−3)	2.7915(−3)	2.6117(−3)	2.0149(−3)
2.600	1.5194(−3)	1.6552(−3)	1.7486(−3)	1.8861(−3)	2.0409(−3)	1.9137(−3)	1.4833(−3)
2.800	1.1225(−3)	1.2219(−3)	1.2902(−3)	1.3904(−3)	1.5037(−3)	1.4125(−3)	1.0991(−3)
3.000	8.3450(−4)	9.0791(−4)	9.5815(−4)	1.0318(−3)	1.1154(−3)	1.0493(−3)	8.1920(−4)
4.000	2.0375(−4)	2.2122(−4)	2.3307(−4)	2.5035(−4)	2.7012(−4)	2.5532(−4)	2.0153(−4)
5.000	5.4060(−5)	5.8629(−5)	6.1708(−5)	6.6182(−5)	7.1317(−5)	6.7579(−5)	5.3665(−5)

TABLE II

The Radiation Current $I_1(\eta z_0, \rho)$ for $z_0 = 1.0$, $\varpi = 0.8$, and $\mu_0 = 1.0$

ρ	$\eta = 0$	$\eta = 0.05$	$\eta = 0.1$	$\eta = 0.2$	$\eta = 0.5$	$\eta = 0.75$	$\eta = 1$
0.001	−6.3217(1)	−5.5479(−1)	8.8949(−2)	3.7952(−1)	4.4737(−1)	4.2432(−1)	2.3563(1)
0.010	−6.0631	−8.0834(−1)	−1.7275(−1)	1.4020(−1)	2.6978(−1)	2.8594(−1)	2.4292
0.100	−4.7074(−1)	−3.7797(−1)	−2.4177(−1)	−6.1321(−2)	9.5914(−2)	1.4392(−1)	2.6473(−1)
0.200	−1.9033(−1)	−1.6963(−1)	−1.3636(−1)	−6.4882(−2)	4.9976(−2)	9.4158(−2)	1.3034(−1)
0.400	−6.5932(−2)	−6.0425(−2)	−5.3057(−2)	−3.4794(−2)	1.6980(−2)	4.4123(−2)	5.6066(−2)
0.600	−3.1437(−2)	−2.8851(−2)	−2.5732(−2)	−1.8235(−2)	6.6958(−3)	2.2312(−2)	2.9825(−2)
0.800	−1.7167(−2)	−1.5717(−2)	−1.4061(−2)	−1.0224(−2)	2.9517(−3)	1.2089(−2)	1.7259(−2)
1.000	−1.0138(−2)	−9.2563(−3)	−8.2814(−3)	−6.0868(−3)	1.4202(−3)	6.9498(−3)	1.0510(−2)
1.200	−6.3099(−3)	−5.7477(−3)	−5.1392(−3)	−3.7982(−3)	7.3166(−4)	4.1954(−3)	6.6435(−3)
1.400	−4.0799(−3)	−3.7097(−3)	−3.3148(−3)	−2.4583(−3)	3.9791(−4)	2.6353(−3)	4.3262(−3)
1.600	−2.7157(−3)	−2.4659(−3)	−2.2023(−3)	−1.6371(−3)	2.2604(−4)	1.7094(−3)	2.8865(−3)
1.800	−1.8494(−3)	−1.6775(−3)	−1.4976(−3)	−1.1153(−3)	1.3307(−4)	1.1382(−3)	1.9652(−3)
2.000	−1.2829(−3)	−1.1627(−3)	−1.0376(−3)	−7.7387(−4)	8.0708(−5)	7.7433(−4)	1.3609(−3)
2.200	−9.0339(−4)	−8.1822(−4)	−7.3004(−4)	−5.4510(−4)	5.0192(−5)	5.3620(−4)	9.5601(−4)
2.400	−6.4424(−4)	−5.8319(−4)	−5.2024(−4)	−3.8882(−4)	3.1891(−5)	3.7686(−4)	6.7990(−4)
2.600	−4.6435(−4)	−4.2017(−4)	−3.7475(−4)	−2.8030(−4)	2.0641(−5)	2.6822(−4)	4.8867(−4)
2.800	−3.3776(−4)	−3.0551(−4)	−2.7245(−4)	−2.0392(−4)	1.3576(−5)	1.9296(−4)	3.5446(−4)
3.000	−2.4762(−4)	−2.2391(−4)	−1.9966(−4)	−1.4952(−4)	9.0570(−6)	1.4010(−4)	2.5917(−4)
4.000	−5.7322(−5)	−5.1778(−5)	−4.6150(−5)	−3.4621(−5)	1.4082(−6)	3.1342(−5)	5.9338(−5)
5.000	−1.4673(−5)	−1.3246(−5)	−1.1803(−5)	−8.8619(−6)	2.6435(−7)	7.8622(−6)	1.5076(−5)

figures of accuracy for these two small values of ρ we found it expedient to use the post-processed results given by Eqs. (68). We note that we typically used $N_a = N_b$ when using Eqs. (68), and since Eq. (68a) is of the form considered in Eq. (79) we used weights and nodes as defined by Eqs. (87) for this calculation. Finally, since Eq. (68b) has the factor $\phi^3(\zeta)$ we developed a special quadrature scheme, in the same spirit as was used to define Eqs. (87), for use in computing $H(z:k)$ from the post-processed expression.

To have some idea about the computational parameters we have used in this work we let z_2 denote the total number of loops of $J_0(x)$ we have considered in Eqs. (73). We also let z_1 denote the number of loops used before the Euler transformation is invoked to accelerate the converge of the Fourier inversion calculation. Continuing we note that in evaluating the integrals over the various loops defined by the zeros of $J_0(x)$ we typically subdivided each integration interval into four subintervals and used, say, N_k Gauss–Legendre quadrature points mapped linearly onto each of the subintervals. Finally to have some idea about the computational parameters we have used to obtain the results given in Tables I and II, we note that we have used, for example, $N = 50$, $N_k = 10$, $z_1 = 75$, and $z_2 = 87$ for all but the smallest two values of ρ. On the other hand, for the smallest two values of ρ we considered, we have used, for example, $N = 20$, $N_k = 50$, $z_1 = 5$, and $z_2 = 17$ with $N_a = 80$ and $N_b = 80$.

Concerning the computational time required to obtain the results reported in Tables I and II, we note, first of all, that only a modest effort has been made to make our FORTRAN implementation of the developed solution efficient as far as speed is concerned. Having said that, we can say that our computations (defined by the parameters mentioned in the foregoing paragraph) that yielded the results given in Tables I and II required 77 minutes (split approximately equally between the two smallest values of ρ and the remaining 18 values of ρ) on a 400 MHz Pentium-based computer. Needless to say, the process of solving an equation of transfer in Fourier-transform space and carrying out a numerical inversion can be considered a computationally intensive task; however, we consider that we have solved well a difficult problem basic to the theory of radiative transfer.

7. CONCLUDING REMARKS

Of course we would like to comment on the way we see this work as a valuable alternative to Ref. [8]. First of all the discrete-ordinates solution developed here does not require any experience with the theory of singular-integral equations, and so we consider this work more readily accessible to many workers in the field of radiative transfer. In addition the solution developed here is not subdivided into three components (the uncollided, the once collided, and the rest), as was done earlier [8], and so some of the analysis is simpler.

While we have not attempted to evaluate our solution for the case of a non-normally incident beam, it is clear that our solution of the G problem requires very little modification for that case. The main challenge for the case of non-normal incidence is the task of evaluating two-dimensional Fourier-inversion integrals; Refs. [9, 10] discuss this case for a semi-infinite ($z_0 \to \infty$) medium.

We consider it worthwhile to mention that we have seen again in this work how very important the choice of the quadrature scheme used to solve the equation of transfer can be. In this regard, we note that in addition to the special-purpose quadrature set summarized by Eqs. (87), we have also used other linear and non-linear maps to define integration schemes. While we generally found good results using these other quadrature schemes for $\rho \geq 0.1$, we found our best results for $\rho < 0.1$ by using Eqs. (87) to define our discrete-ordinates solution of the equation of transfer. Finally, we have seen in this work the interesting (at least to us) situation where one of the quantities, viz. $H(z)$, we wished to compute was well evaluated by using a second special-purpose quadrature scheme in the relevant post-processing expression, i.e., Eq. (68b).

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